*Standard Forms of Certain Types of Peirce Algebras.**

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I. INTRODUCTION.

1. The determination of general laws for the relationships of numbers in an algebra of order r has not progressed very far, especially as regards PEIRCE algebras. By PEIRCE algebra is meant an algebra containing an idempotent unit η, which is the modulus, all other expressions being nilpotent unless they contain a term of the form $a\eta$, where a is a scalar coefficient. The reduction of algebras in general must depend on such laws of structure of an algebra, and it seems that even in the simpler cases, after we have reduced to forms that have a comparatively simple multiplication table, the complete exhaustion of all the information which can possibly be obtained by using the law of associativity leaves nevertheless a number of arbitrary parameters which can only be removed by linear transformation of the units, if removable at all. The further determination of individual types becomes then somewhat a matter of personal choice. The present paper does not consider this question, which has been touched upon elsewhere. It seeks only to reduce certain particular cases to their simplest forms, thus extending the present narrow list considerably.

2. It is known that, for any PEIRCE algebra, we may take any one of the nilpotent expressions to be a unit, called the *adjunct unit*,† represented by e_{11}, determine then a set of expressions called the *base*, defined by units η, e_{20}, e_{30}, $\ldots$, e_{m0}. Any expression of the algebra is then linearly expressible in terms of

$$\eta,\ e_{i0},\ e_{11},\ e_{i0}\,e_{11},\ e_{11}^2,\ e_{i0}\,e_{11}^2,\ \ldots,\ e_{i0}\,e_{11,}^{\mu_i-1},\ \ldots,\ e_{11}^{\mu_1-1}.$$

It is preferable of course to choose for e_{11} a number which will give μ_1 as high a

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* Read before the Chicago Section of the American Mathematical Society, Dec. 30, 1907.

† SHAW: On Nilpotent Algebras, *Trans. Amer. Math. Soc.* (1903), 4, 405–422.

value as will any other expression in the algebra. In that event, $r - \mu_1$ is called the *deficiency* of the algebra.

3. SCHEFFERS* and others have shown that the units may be *regularised;* that is, put into an order $e_1, \ldots, e_i, \ldots, e_j, \ldots$ such that $e_i e_j$ and $e_j e_i$ are of the form $\Sigma \gamma_{ijk} e_k$ where $k > i$, and $k > j$. The theorem quoted in § 2 implies this.

4. PEIRCE† showed that when the *deficiency is zero,* the algebra is expressible in terms of the units $\eta, e_{11}, \ldots, e_{11}^{r-1}$, or, in a more convenient notation, $\eta, i, i^2, \ldots, i^{r-1}$. He showed further that when the *deficiency is unity,* the algebra is defined by the units $\eta, i, j, \ldots, j^{r-2}$, where

$$ij = 0, \quad i^2 = aj^{r-2}, \quad ji = bj^{r-2}.$$

We find here four types not reducible into each other, namely

I. $a = b = 0.$
II. $a = 0,\ b = 1.$
III. $a = 1,\ b = 0.$
IV. $a = b = 1.$

5. The cases of *deficiency two* were reduced in full by STARKWEATHER.‡ There are three types, each with numerous sub-types. They are as follows.

When $r > 6$.

(1) *Type* $(\eta, i, i^2, j, j^2, \ldots, j^{r-3})$.

(11) $i^3 = j^{r-3}$; $ij = 0 = ji$.

(12) $i^3 = 0$; $ij = 0 = ji$.

(13) $i^3 = j^{r-3}$; $ij = 0$; $ji = 2j^{r-3}$; $ji^2 = 0$.

(14) $i^3 = 0$; $ij = 0$; $ji = 2j^{r-3}$; $ji^2 = 0$.

(2) *Type* $(\eta, i, j, ij, j^2, \ldots, j^{r-3})$.

(21) $i^2 = j^{r-4}$; $i^3 = 0$; $ji = ij$; $i^2j = iji = ji^2 = j^{r-3}$.

(22) $i^2 = j^{r-4}$; $ji = ij + 2j^{r-3}$; $i^2j = j^{r-3} = iji = ji^2$.

(23) $i^2 = j^{r-4}$; $ji = ij + 2j^{r-4} + 2cj^{r-3}$; $i^2j = j^{r-3} = iji = ji^2$; $jij = 2j^{r-3}$.

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* SCHEFFERS: *Mathematische Annalen* (1891), 39, 293–390.

† B. PEIRCE: AMER. JOUR. OF MATH. (1881), 4, 97–192.

‡ STARKWEATHER: AMER. JOUR. OF MATH. (1899), 21, 369–386; (1901), 23, 378–402.

In this sub-type $c = 0$ when $r \neq 8$.

(24) $i^2 = j^{r-4} + j^{r-3}$; $ji = -ij - 2j^{r-4}$; $i^2j = j^{r-3}$; $iji = -j^{r-3}$; $ji^2 = j^{r-3}$; $jij = -2j^{r-3}$.

(25) $i^2 = j^{r-4}$; $ji = -ij$; $i^2j = j^{r-3}$; $iji = -j^{r-3}$; $ji^2 = j^{r-3}$.

(26) $i^2 = hj^{r-3}$; $ji = \frac{c}{2-c} ij + 2(2-c)j^{r-4}$; $iji = 0 = i^2j = ji^2$; $jij = 2(2-c)j^{r-3}$; $j^2i = 4j^{r-3}$.

In this sub-type $h = 0$ or 1 when $r \neq 7$.

(27) $i^2 = j^{r-3}$; $ji = -ij$; $i^2j = 0 = ji^2 = iji = j^2i = jij$.

(28) $i^2 = hj^{r-3}$; $ji = ij + 2j^{r-3}$; $i^2j = 0 = jij = iji = ji^2 = j^2i$.

In this sub-type $h = 0$ or 1 when $r \neq 7$.

(29) $i^2 = j^{r-3}$; $ji = dij$; $i^2j = 0 = iji = ji^2 = j^2i = jij$.

(2α) $i^2 = 0$; $ji = dij$; $i^2j = 0 = iji = ji^2 = j^2i = jij$.

(2β) $i^2 = ij + j^{r-3}$; $ji = 0$; $i^2j = iji = ji^2 = 0 = j^2i = jij = i^3$.

(2γ) $i^2 = ij$; $ji = 0$; $i^2j = iji = ji^2 = 0 = j^2i = jij = i^3$.

(2ε) $i^2 = j^{r-3}$; $ji = 2j^{r-3}$.

(2ζ) $i^2 = j^{r-3}$; $ji = 0$.

(2η) $i^2 = 0 = ji$.

(3) *Type* $(\eta, i, j, k, \ldots, k^{r-3})$.

(31) $i^2 = k^{r-3}$; $ij = ji = 0 = ki = kj = j^2 = jk$.

(32) $i^2 = 0 = ji = ij = ki = ik = jk = kj = j^2$.

(33) $i^2 = 0 = j^2 = ik = ki = jk = kj$; $ij = gk^{r-3}$; $ji = k^{r-3}$.

(34) $i^2 = 0 = j^2$; $ij = k^{r-3} = ji$; $ki = 2k^{r-3} = kj$.

(35) $i^2 = 0 = j^2$; $ij = k^{r-3} = ji$; $kj = 2k^{r-3}$; $ki = 0$.

(36) $i^2 = k^{r-3}$; $j^2 = 0$; $ij = -k^{r-3}$; $ji = k^{r-3}$; $ki = 0 = kj$.

(37) $i^2 = k^{r-3}$; $j^2 = 0$; $ij = 0 = ji$; $ki = 0$; $kj = 2k^{r-3}$.

(38) $i^2 = k^{r-3}$; $j^2 = 0 = ij = ji$; $ki = 2k^{r-3}$; $kj = 0$.

(39) $i^2 = 0 = j^2 = ij = ji = ki$; $kj = 2k^{r-3}$.

The forms which these algebras take when $r = 4$, 5, or 6 appear in SCHEFFERS' * and STARKWEATHER'S † lists, and may be found also in SHAW'S ‡ "Synopsis."

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* SCHEFFERS: *Math. Ann.* (1891), 39, 293–390.

† STARKWEATHER: AMER. JOUR. OF MATH. (1901), 23, 378–402.

‡ SHAW: "Synopsis of Linear Associative Algebra," Carnegie Institution of Washington, D. C., pp. 103, 105, 106.

6. The theorem mentioned in § 2 may be expressed more definitely by representing the generators in terms of certain ideal units denoted by λ_{ijk}, thus:

$$\begin{aligned} e_{f0} &= \Sigma b_{gh}\,\lambda_{gh0} + \Sigma a_{ijk}\,\lambda_{ijk}, && (i, j = 1, 2, \ldots, m) \\ e_{11} &= \lambda_{111} + \Sigma a'_{ijk}\,\lambda_{ijk}, && (i, j = 1, 2, \ldots, m) \\ \eta &= \lambda_{110} + \lambda_{220} + \ldots + \lambda_{mm0}, \end{aligned}$$

where $\mu_i > k > 1$, and $k \geqq \mu_i - \mu_j$, $g > h$. Further, the coefficients b_{gh} are so chosen that if the terms λ_{ijk} be cut off from the expression for e_{f0}, giving

$$e'_{f0} = \Sigma b_{gh}\,\lambda_{gh0},$$

then these units define an associative algebra. The ideal units λ_{ijk} satisfy the laws

$$\lambda_{ijk}\,\lambda_{i'j'k'} = \vartheta_{ji'}\,c\lambda_{ij'\,k+k'},$$

where $c = 1$ when $\mu_i > k + k' \geqq \mu_i - \mu_{j'}$, $k + k' \geqq 0$; otherwise $c = 0$. Also $\vartheta_{ji'} = 1$ if $j = i'$, otherwise $\vartheta_{ji'} = 0$.

In this notation, and starting from this theorem, we may produce the complete set of sub-types of an algebra whose type is assigned. It is purposed to study a few types by this method, both for the results obtained and to show the utility of the method.

II. THE TYPE $(\eta, i, \ldots, i^m, j, \ldots, j^{r-m-1})$.

1. It is implied that $ij = 0$. Since $j^{r-m} = 0$, $\mu_1 = r - m$; since $ij = i^2j = \ldots = i^mj = 0$, $\mu_2 = \mu_3 = \ldots = \mu_{m+1} = 1$. Hence we will find no other forms of λ than $\lambda_{210}, \lambda_{320}, \lambda_{430}, \ldots, \lambda_{m+1, m, 0}, \lambda_{111}, \ldots, \lambda_{1, 1, r-m-1}, \lambda_{1, t, r-m-1}$ $(t = 2, \ldots, m+1)$.

2. If for convenience we write only the subscripts, omitting λ, then

$$i = (210) + (320) + \ldots + (m+1, m, 0) + \sum_{t=2}^{m+1} a_t\,(1, t, r - m - 1),$$

$$j = (111) + \sum_{t=2}^{m+1} b_t\,(1, t, r - m - 1).$$

Thus

$$i^2 = (310) + \ldots + (m+1, m-1, 0) + \sum_{t=2}^{m+1} a_t\,(1, t-1, r - m - 1).$$

For $t = 2$, however, $(1, t-1, r-m-1)$ becomes $(1, 1, r-m-1)$ and i^2 would contain a term $a_2\,j^{r-m-1}$, which is not possible if $m > 1$. The type for $m = 1$ is included above in I, and need not be discussed again. Hence $a_2 = 0$. Likewise, from $i^3, \ldots, i^m$ we have $a_t = 0$ for $t = 2, 3, \ldots, m$.

Therefore

$$
\begin{aligned}
i \ \ &= (210) + (320) + \dots + (m+1,\, m,\, 0) + a(1,\, m+1,\, r-m-1);\\
i^2 \ \ &= (310) + (420) + \dots + (m+1,\, m-1,\, 0) + a(1,\, m,\, r-m-1);\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots;\\
i^m \ \ &= (m+1,\, 1,\, 0) + a(1,\, 2,\, r-m-1);\\
i^{m+1} &= a(1,\, 1,\, r-m-1).
\end{aligned}
$$

3. Again,

$$j^2 = (112), \dots, j^{r-m-1} = (1,\, 1,\, r-m-1);$$

$$ji = \sum_{t=2}^{m+1} b_t\,(1,\, t-1,\, r-m-1).$$

But this gives $ji = b_2 j^{r-m-1}$, and $b_t = 0$ for $t > 2$. We have therefore

$$j = (111) + b(1,\, 2,\, r-m-1).$$

4. Hence we may suppose finally for all algebras of this type that

$$i^{m+1} = a j^{r-m-1};$$

$$ij = 0; \quad ji = b j^{r-m-1}.$$

If $a \neq 0$, $b = 0$, we may choose $i_1 = i a^{-\frac{1}{m+1}}$, whence

$$i_1^{m+1} = j^{r-m-1}, \quad j i_1 = 0.$$

If $a = 0$, $b \neq 0$, we may take $j_1 = j b^{\frac{1}{r-m-2}}$, whence

$$i^{m+1} = 0, \quad i j_1 = 0, \quad j_1 i = j_1^{r-m-1}.$$

If $a \neq 0 \neq b$, we may take $i_1 = i\left(a^{-1} b^{\frac{r-m-1}{r-m-2}}\right)^{\frac{r-m-2}{m(r-m-2)-1}}$,

$$j_1 = j(a^{-1}\, b^{m+1})^{\frac{1}{m(r-m-2)-1}}; \text{ whence}$$

$$i_1^{\ \ +1} = j_1^{r-m-1}, \ i_1 j_1 = 0, \ j_1 i_1 = j_1^{r-m-1}.$$

5. We have then four sub-types, given by the equations:

$$
\begin{aligned}
&(1) \quad i^{m+1} = j^{r-m-1}, \quad ji = 0.\\
&(2) \quad i^{m+1} = 0, \qquad\qquad ji = j^{r-m-1}.\\
&(3) \quad i^{m+1} = j^{r-m-1} = ji.\\
&(4) \quad i^{m+1} = 0 = ji.
\end{aligned}
$$

The types worked out by BENJAMIN PEIRCE for *deficiency unity* are thus extended to all the types for the class represented by the symbol

$$(\eta,\, i,\, i^2,\, \dots,\, i^m,\, j,\, j^2,\, \dots,\, j^{r-m-1}).$$

III. THE TYPE $(\eta, i, i^2, \ldots, i^m, j, ij, j^2, \ldots, j^{r-m-2})$.

1. We have now $\mu_1 = r - m - 1$, $\mu_2 = 2$, $\mu_3 = \mu_4 = \ldots = \mu_{m+1} = 1$. Thus

$$i = (210) + \ldots + (m+1, m, 0) + \sum_{t=2}^{m+1} a_t (2, t, 1) + b(1, 2, r-m-3) + \sum_{v=3}^{m+1} c_v (1, v, r-m-2);$$

$$j = (111) + \sum_{t=2}^{m+1} d_t (2, t, 1) + f(1, 2, r-m-3) + \sum_{v=3}^{m+1} g_v (1, v, r-m-2).$$

2. Thus

$$i^2 = (310) + \ldots + (m+1, m-1, 0) + \sum_{t=2}^{m+1} a_t (2, t-1, 1) + b(1, 1, r-m-3) + \sum_{v=3}^{m+1} c_v (1, v-1, r-m-2).$$

Now i^2 is independent of ij or j^{r-m-3}, so that $a_2 = 0 = b$ unless $m = 1$. Since the deficiency would thus be 2, this case has been considered. Likewise we find

$$a_t = 0, \qquad t < m+1; \qquad c_v = 0 \text{ if } v < m+1.$$

Therefore

$$i = (210) + \ldots + (m+1, m, 0) + a(2, m+1, 1) + c(1, m+1, r-m-2);$$
$$\ldots\ldots\ldots\ldots;$$
$$i^{m+1} = a(211) + c(1, 1, r-m-2);$$
$$ij = (211).$$

3. Again,

$$j^2 = (112) + f(1, 2, r-m-2) + \sum_{t=2}^{m+1} d_t f(1, t, r-m-2);$$
$$j^3 = (113);$$
$$\ldots\ldots;$$
$$j^{r-m-2} = (1, 1, r-m-2);$$
$$ji = \sum_{t=2}^{m+1} d_t (2, t-1, 1) + f(1, 1, r-m-3) + fa(1, m+1, r-m-2) + \sum_{v=3}^{m+1} g_v (1, v-1, r-m-2) = d_2 (211) + f(1, 1, r-m-3).$$

Therefore

$$d_t = 0 \text{ for } t > 2, \quad fa = 0, \quad g_v = 0 \text{ for } v = 3, \ldots, m+1,$$

and

$$j = (111) + d(221) + f(1, 2, r-m-3),$$

with either

$$f = 0 \text{ or } a = 0.$$

There are then two subdivisions:

(1) $i = (210) + \ldots + (m+1, m, 0) + c(1, m+1, r-m-2)$,
$j = (111) + d(221) + f(1, 2, r-m-3)$.

(2) $i = (210) + \ldots + (m+1, m, 0) + a(2, m+1, 1) + c(1, m+1, r-m-2)$,
$j = (111) + d(221)$.

In (1), if $c \neq 0$ we may take it equal to 1, and if $f \neq 0$ we may take $f = 1$. We may proceed likewise with a and c in (2). Thus we have the sub-types of this type ($m > 1$):

$$\left.\begin{array}{lll}
(1) & i^{m+1} = 0, & ji = dij; \\
(2) & i^{m+1} = j^{r-m-2}, & ji = dij; \\
(3) & i^{m+1} = ij, & ji = dij; \\
(4) & i^{m+1} = ij + j^{r-m-2}, & ji = dij; \\
(5) & i^{m+1} = 0, & ji = dij + j^{r-m-3}; \\
(6) & i^{m+1} = j^{r-m-2}, & ji = dij + j^{r-m-3}.
\end{array}\right\} d \text{ arbitrary.}$$

IV. THE TYPE $(\eta, i, j, \ldots, j^{m-1}, k, \ldots, k^{r-m-1})$.

(a) *When Neither i^2 nor ji Contains j^{m-1}.*

1. Here

$$\mu_1 = r - m, \quad \mu_2 = \mu_3 = \ldots = \mu_{m+1} = 1.$$

Hence

$$i = (210) + \ldots + \sum_{t=2}^{m+1} a_t (1, t, r-m-1);$$

$$j = (310) + \ldots + (m+1, m, 0) + \sum_{u=2}^{m+1} b_u (1, u, r-m-1);$$

$$k = (111) + \sum_{v=2}^{m+1} c_v (1, v, r-m-1).$$

Hence

$$i^2 = a_2 (1, 1, r-m-1);$$

$$ij = a_3 (1, 1, r-m-1) + a_4 (1, 3, r-m-1) + \ldots + a_{m+1} (1, m, r-m-1).$$

Therefore

$$a_t = 0 \quad \text{for} \quad t > 3.$$

Again,

$$ji = b_2 (1, 1, r-m-1);$$

$$j^2 = (410) + (530) + \ldots + (m+1, m-1, 0) + b_3 (1, 1, r-m-1) + b_4 (1, 3, r-m-1) + \ldots + b_{m+1} (1, m, r-m-1).$$

Hence
$$b_3 = 0, \quad \text{and likewise} \quad b_4 = 0 = b_5 = \ldots\ldots = b_m.$$

Again,
$$ki = c_2(1, 1, r - m - 1);$$
$$kj = c_3(1, 1, r - m - 1) + c_4(1, 3, r - m - 1) + \ldots\ldots;$$
and
$$c_v = 0 \quad \text{for} \quad v > 3.$$

2. Thence
$$i = (210) + a_2(1, 2, r - m - 1) + a_3(1, 3, r - m - 1);$$
$$j = (310) + \ldots\ldots + (m + 1, m, 0) + b_2(1, 2, r - m - 1)$$
$$+ b_{m+1}(1, m + 1, r - m - 1);$$
$$k = (111) + c_2(1, 2, r - m - 1) + c_3(1, 3, r - m - 1).$$

The defining equations of the algebra become
$$i^2 = a_2 k^{r-m-1}; \qquad j^m = b_{m+1} k^{r-m-1}; \qquad ij = a_3 k^{r-m-1};$$
$$ji = b_2 k^{r-m-1}; \qquad ki = c_2 k^{r-m-1}; \qquad kj = c_3 k^{r-m-1}.$$

If $a_2 \neq 0$ we may take $i = a^{\frac{1}{2}} i_1$, which amounts to supposing $a_2 = 1$. Likewise, if $b_{m+1} \neq 0$ we may take $b_{m+1} = 1$. In case $a_2 \neq 0$, $b_{m+1} \neq 0$, $a_3 \neq 0$, we may change i, j, k into such multiples that
$$i^2 = k^{r-m-1}, \qquad j^m = k^{r-m-1}, \qquad ij = k^{r-m-1}.$$

(b) When i^2 Contains j^{m-1}, but ji Does not.

3. In this case
$$i = (210) + (m + 1, 2, 0) + \sum_{t=2}^{m+1} a_t(1, t, r - m - 1);$$
$$j = (310) + \ldots\ldots + (m + 1, m, 0) + \sum_{u=2}^{m+1} b_u(1, u, r - m - 1);$$
$$k = (111) + \sum_{v=2}^{m+1} c_v(1, v, r - m - 1).$$

Hence
$$i^2 = (m + 1, 1, 0) + a_2(1, 1, r - m - 1);$$
$$ij = a_3(1, 1, r - m - 1) + \ldots\ldots;$$
and
$$a_t = 0 \quad \text{for} \quad t > 3.$$

Also
$$ji = b_2(1, 1, r - m - 1) + b_{m+1}(1, 2, r - m - 1);$$
$$j^2 = (410) + \ldots\ldots + (m + 1, m - 1, 0) + b_3(1, 1, r - m - 1)$$
$$+ \ldots\ldots + b_{m+1}(1, m, r - m - 1).$$

Hence
$$b_u = 0 \quad \text{for} \quad u > 2.$$

Again,

$$ki = c_2(1, 1, r-m-1) + c_{m+1}(1, 2, r-m-1);$$
$$kj = c_3(1, 1, r-m-1) + \ldots.;$$

and

$$c_v = 0 \quad \text{for} \quad v > 3.$$

4. We have therefore finally

$$i^2 = j^{m-1} + a_2 k^{r-m-1}, \qquad j^m = 0, \qquad ij = a_3 k^{r-m-1},$$
$$ji = b_2 k^{r-m-1}, \qquad ki = c_2 k^{r-m-1}, \qquad kj = c_3 k^{r-m-1}.$$

(c) When ji Contains j^{m-1} but i^2 Does not.

5. $$i = (210) + \sum_{t=2}^{m+1} a_t(1, t, r-m-1);$$
$$j = (310) + \ldots. + (m+1, 2, 0) + (m+1, m, 0) + \sum_{u=2}^{m+1} b_u(1, u, r-m-1);$$
$$k = (111) + \sum_{v=2}^{m+1} c_v(1, v, r-m-1);$$
$$i^2 = a_2(1, 1, r-m-1);$$
$$ij = a_3(1, 1, r-m-1) + \ldots. + a_{m+1}(1, m, r-m-1)$$
$$+ a_{m+1}(1, 2, r-m-1).$$

Hence

$$a_t = 0 \quad \text{for} \quad t > 3.$$
$$j^2 = (410) + \ldots. + (m+1, m-1, 0) + b_3(1, 1, r-m-1)$$
$$+ b_4(1, 3, r-m-1) + \ldots. + b_{m+1}(1, 2, r-m-1);$$
$$ji = (m+1, 1, 0) + b_2(1, 1, r-m-1).$$

Therefore

$$b_u = 0 \quad \text{for} \quad m+1 > u > 2.$$

Also

$$jij = 0; \quad \text{hence} \quad b_{m+1} = 0.$$

6. Hence, finally,

$$i^2 = a_2 k^{r-m-1}; \qquad j^m = 0; \qquad ij = a_3 k^{r-m-1};$$
$$ji = b_2 k^{r-m-1} + j^{m-1}; \qquad ki = c_2 k^{r-m-1}; \qquad kj = c_3 k^{r-m-1}.$$

(d) When Both i^2 and ji Contain j^{m-1}.

7. The analysis leads to the equations

$$i^2 = j^{m-1} + a_2 k^{r-m-1}; \qquad j^m = 0; \qquad ij = a_3 k^{r-m-1};$$
$$ji = b_2 k^{r-m-1} + j^{m-1}; \qquad ki = c_2 k^{r-m-1}; \qquad kj = c_3 k^{r-m-1}.$$

V. THE TYPE $(\eta, i, j, ij, j^2, ij^2, j^3, \ldots, j^{r-4})$.

1. In this type $\mu_1 = r - 3$, $\mu_2 = 3$; hence

$$i = (210) + b(221) + c(222) + d_1(1, 2, r-6) + d_2(1, 2, r-5) + d_3(1, 2, r-4);$$
$$j = (111) + g(221) + h(222) + f_1(1, 2, r-6) + f_2(1, 2, r-5) + f_3(1, 2, r-4).$$

Therefore

$$ij = (211) + bg(222) + d_1g(1, 2, r-5) + (d_2g + d_1h)(1, 2, r-4);$$
$$ij^2 = (211) + d_1g(1, 2, r-4);$$
$$j^2 = (112) + g^2(222) + f_1(g+1)(1, 2, r-5) + (f_2 + f_1h + f_2g)(1, 2, r-4);$$
$$ij^3 = (113) + f_1(g^2 + g + 1)(1, 2, r-4);$$
$$j^4 = (114).$$

(*a*) *When* $r > 9$.

2. $$i^2 = b(211) + c(212) + b^2(222) + d_1(1, 1, r-6) + d_2(1, 1, r-5) + d_3(1, 1, r-4) + bd_1(1, 2, r-5) + (bd_2 + cd_1)(1, 2, r-4)$$
$$= b(211) + b^2g(222) + bd_1g(1, 2, r-5) + b(d_2g + d_1h)(1, 2, r-4) + c(212) + cd_1g(1, 2, r-4) + d_1(1, 1, r-6) + d_2(1, 1, r-5) + d_3(1, 1, r-4).$$

Thus

$$b^2 = b^2g, \qquad bd_1 = bd_1g, \qquad bd_2 + cd_1 = bd_2g + bd_1h + cd_1g.$$

3. Again,

$$ji = g(211) + h(212) + f_1(1, 1, r-6) + f_2(1, 1, r-5) + f_3(1, 1, r-4) + bg(222) + (d_1 + bf_1)(1, 2, r-5) + (d_2 + cf_1)(1, 2, r-4)$$
$$= g(211) + bg^2(222) + d_1g^2(1, 2, r-5) + (d_2g^2 + 2d_1gh)(1, 2, r-4) + h(212) + f_1(1, 1, r-6) + f_2(1, 1, r-5) + f_3(1, 1, r-4).$$

Thus

$$bg^2 = bg, \qquad d_1 + bf_1 = d_1g^2, \qquad d_2 + f_1c = d_2g^2 + 2d_1gh.$$

4. Next,

$$j^2i = g^2(212) + f_1(g+1)(1, 1, r-5) + (f_2 + f_1h + f_2g)(1, 1, r-4) + (d_1 + bf_1(g+1))(1, 2, r-4).$$

Therefore

$$d_1(g^3 - 1) = bf_1(g+1).$$

5. Collecting these results we find the following sub-types:

$$
\begin{array}{llllll}
(1) & b=0, & c=0, & g\neq\pm 1, & d_1=0, & d_2=0.\\
(2) & b=0, & c=0, & g=\pm 1, & d_1=0. & \\
(3) & b=0, & c=0, & g=1, & h=0. & \\
(4) & b=0, & c\neq 0, & g=1, & 2d_1h-cf_1=0. & \\
(5) & b=0, & c\neq 0, & g\neq 1, & d_1=0, & f_1=0.\\
(6) & b\neq 0, & g=1, & d_1=0, & f_1=0. & \\
(7) & b\neq 0, & g=1, & h=0, & f_1=0. &
\end{array}
$$

The resulting equations are

$$
\begin{aligned}
(1)\quad & i^2=d_3j^{r-4}; \quad ji=gij+hij^2+f_1j^{r-6}+f_2j^{r-5}+f_3j^{r-4};\\
& j^2i=g^2ij^2+f_1(g+1)j^{r-5}+(f_2+f_1h+f_2g)j^{r-4};\\
& j^3i=f_1(g+1)j^{r-4};\\
& jij=gij^2+f_1j^{r-5}+f_2j^{r-4};\\
& jij^2=f_1j^{r-4};\\
& j^2ij=f_1(g+1)j^{r-4}.
\end{aligned}
$$

$$
\begin{aligned}
(2_1)\quad & i^2=d_2j^{r-5}+d_3j^{r-4}; \quad ji=ij+hij^2+f_1j^{r-6}+f_2j^{r-5}+f_3j^{r-4};\\
& j^2i=ij^2+2f_1j^{r-5}+(2f_2+f_1h)j^{r-4}; \quad ji^2=d_2j^{r-4}=i^2j;\\
& j^3i=2f_1j^{r-4};\\
& jij=ij^2+f_1j^{r-5}+f_2j^{r-4};\\
& jij^2=f_1j^{r-4};\\
& j^2ij=2f_1j^{r-4}.
\end{aligned}
$$

$$
\begin{aligned}
(2_2)\quad & i^2=d_2j^{r-5}+d_3j^{r-4}; \quad ji=-ij+h\cdot ij^2+f_1j^{r-6}+f_2j^{r-5}+f_3j^{r-4};\\
& j^2i=ij^2+f_1hj^{r-4}; \quad j^3i=0;\\
& jij=-ij^2+f_1j^{r-5}+f_2j^{r-4}; \quad jij^2=f_1j^{r-4}; \quad j^2ij=0;\\
& ji^2=d_2j^{r-4}=i^2j.
\end{aligned}
$$

$$
\begin{aligned}
(3)\quad & i^2=d_1j^{r-6}+d_2j^{r-5}+d_3j^{r-4}; \quad ji=ij+f_1j^{r-6}+f_2j^{r-5}+f_3j^{r-4};\\
& j^2i=ij^2+2f_1j^{r-5}+2f_2j^{r-4}; \quad j^3i=3f_1j^{r-4}; \quad jij=ij^2\\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +f_1j^{r-5}+f_2j^{r-4};\\
& jij^2=f_1j^{r-4}; \quad j^2ij=2f_1j^{r-4}; \quad j^3i=2f_1j^{r-4};\\
& ji^2=d_1j^{r-5}+d_2j^{r-4}=i^2j; \quad j^2i^2=d_1j^{r-4}=i^2j^2.
\end{aligned}
$$

$$
\begin{aligned}
(4)\quad & i^2=ij^2+d_1j^{r-6}+d_2j^{r-5}+d_3j^{r-4}; \quad ji=ij+hij^2\\
& \qquad\qquad\qquad\qquad\qquad +\frac{2d_1h}{c}j^{r-6}+f_2j^{r-5}+f_3j^{r-4};
\end{aligned}
$$

$$jij = ij^2 + \frac{2d_1h}{c}j^{r-5} + f_2j^{r-4}; \quad jij^2 = \frac{2d_1h}{c}j^{r-4};$$

$$j^2i = ij^2 + \frac{4d_1h}{c}j^{r-5} + 2f_2j^{r-4}; \quad j^2ij = \frac{4d_1h}{c}j^{r-4}; \quad j^3i = \frac{4d_1h}{c}j^{r-4};$$

$$ji^2 = d_1j^{r-5} + d_2j^{r-4} = i^2j; \quad j^2i^2 = d_1j^{r-4} = i^2j^2.$$

(5) $$i^2 = cij^2 + d_2j^{r-5} + d_3j^{r-4}; \quad ji = gij + hij^2 + f_2j^{r-5} + f_3j^{r-4};$$
$$jij = gij^2 + f_2j^{r-4}; \quad j^2i = g^2ij^2 + (g+1)f_2j^{r-4};$$
$$j^2ij = 0; \quad jij^2 = 0; \quad j^3i = 0; \quad ji^2 = d_2j^{r-4} = i^2j.$$

(6) $$i^2 = bij + cij^2 + d_2j^{r-5} + d_3j^{r-4}; \quad ji = ij + hij^2 + f_2j^{r-5} + f_3j^{r-4};$$
$$jij = ij^2 + f_2j^{r-4}; \quad jij^2 = 0; \quad i^2j = bij^2 + d_2j^{r-4} = iji;$$
$$ji^2 = bij^2 + (bf_2 + d)j^{r-4};$$
$$j^2i = ij^2 + 2f_2j^{r-4}; \quad j^2ij = 0 = j^3i; \quad j^2i^2 = 0.$$

(7) $$i^2 = bij + cij^2 + d_1j^{r-6} + d_2j^{r-5} + d_3j^{r-4}; \quad ji = ij + f_2j^{r-5} + f_3j^{r-4};$$
$$jij = ij^2 + f_2j^{r-4}; \quad jij^2 = 0 = j^2ij = j^3i;$$
$$j^2i = ij^2 + 2f_2j^{r-4}; \quad i^2j = bij^2 + d_1j^{r-5} + d_2j^{r-4} = iji;$$
$$ji^2 = bij^2 + d_1j^{r-5} + (bf_2 + d_2)j^{r-4};$$
$$i^3 = b^2ij^2 + bd_1j^{r-5} + bd_2j^{r-4}; \quad ji^3 = i^3j = bd_1j^{r-4}; \quad i^2j^2 = d_1j^{r-4} = j^2i^2.$$

(*b*) *When* $r = 6$.

6. We have some modifications in the general formulæ. Thus

$$\begin{aligned} i &= (210) + b(221) + c(222) + d(122); \\ j &= (111) + g(221) + h(222) + n(122); \\ ij &= (211) + (n + bg)(222); \\ j^2 &= (112) + g^2(222); \\ ij^2 &= (212). \end{aligned}$$

We find easily

$$i^2 = b(211) + c(212) + (b^2 + d)(222) + d(112).$$

Hence

$$b(n + bg) = -d(g^2 - 1) + b^2.$$

Again

$$ji = g(211) + h(212) + bg(222) + n(112).$$

Hence

$$g(n + bg) + ng^2 = bg, \quad \text{or} \quad (b + n)g^2 = (b - n)g.$$

The sub-types are then

(1) $$g = 0, \quad d = bn - b^2.$$
$$i^2 = bij + cij^2 + (bn - b^2)j^2; \quad ji = hij^2 + nj^2.$$
$$i^3 = b^2ij^2.$$

(2) $g = 1, \quad n = 0.$
$i^2 = bij + cij^2 + dj^2; \quad ji = ij + hij^2.$
$i^3 = b^2ij^2.$

(3) $g = -1, \quad b = 0.$
$i^2 = cij^2 + dj^2; \quad ji = -ij + hij^2 + nj^2.$
$i^3 = 0.$

(4) $g \neq 0, \quad g \neq \pm 1.$
$i^2 = bij + cij^2 + \frac{bng}{g^2-1} j^2; \quad ji = gij + hij^2 + nj^2.$
$i^3 = b^2ij^2.$

(c) *When* $r = 7$.

7. In this case

$$\begin{aligned}
i &= (210) + a(221) + b(222) + c(122) + d(123);\\
j &= (111) + f(221) + g(222) + h(122) + l(123);\\
ij &= (211) + (h + af)(222) + cf(123);\\
j^2 &= (112) + f^2(222) + (f+1)h(123);\\
ij^2 &= (212);\\
j^3 &= (113).
\end{aligned}$$

Then

$$\begin{aligned}
i^2 &= a(211) + b(212) + (a^2 + c)(222) + c(112) + d(113) + ac(123)\\
&= a(211) + a(h + af)(222) + acf(123) + c(112) + cf^2(222)\\
&\qquad + c(f+1)h(123) + b(212) + d(113).
\end{aligned}$$

Hence

$$a^2 + c = cf^2 + a^2f + ah, \qquad (h + a)cf + ch = ac.$$

Again,

$$ji = f(211) + g(212) + af(222) + (ah + c)(123) + h(112) + l(113).$$

Hence

$$\begin{aligned}
fh + af^2 + f^2h &= af,\\
cf^2 + (f+1)h^2 &= c + ah.
\end{aligned}$$

The sub-types are therefore

(1) $f = 0, \quad a = h, \quad c = 0. \qquad i^2 = aij + dj^3; \quad ji = gij^2 + lj^3.$

(2) $f \neq 0, \quad a = 0 = c = h. \qquad i^2 = dj^3; \quad ji = fij + gij^2 + lj^3.$

(3) $f = \frac{a-h}{a+h}, \quad a \neq h, \quad c = \frac{1}{4}(h^2 - a^2).$
$i^2 = aij + \frac{1}{4}(h^2 - a^2)j^2 + dj^3.$
$ji = \frac{a-h}{a+h} ij + gij^2 + lj^3.$

(*d*) *When* $r=8$.

8. We find, in the same way as before,

$$
\begin{aligned}
i &= (210) + a(221) + b(222) + c(122) + d(123) + e(124);\\
j &= (111) + f(221) + g(222) + h(122) + k(123) + l(124);\\
ij &= (211) + (af+h)(222) + cf(123) + (cg+df)(124);\\
j^2 &= (112) + f^2(222) + (1+f)h(123) + (1+f)k(124);\\
ij^2 &= (212) + cf^2(124);\\
j^3 &= (113) + h(1+f+f^2)(124);\\
i^2 &= a(211) + b(212) + c(112) + (a^2+c)(222) + d(113)\\
&\qquad + e(114) + ac(123) + (ad+bc)(124)\\
&= a(211) + (a^2f+ah)(222) + acf(123) + (acg+adf)(124)\\
&\qquad + b(212) + bcf^2(124) + c(112) + cf^2(222) + (ch+cfh)(123)\\
&\qquad + (ck+cfk)(124) + d(113) + dh(f^2+f+1)(124) + e(114).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&a^2(f-1) + c(f^2-1) + ah = 0,\\
&ac(f-1) + ch(f+1) = 0,\\
&ad(f-1) + bc(f^2-1) + ck(f+1) + acg + dh(f^2+f+1) = 0.
\end{aligned}
$$

Again

$$
\begin{aligned}
ji &= f(211) + g(212) + h(112) + k(113) + l(114) + af(222)\\
&\qquad + (c+ah)(123) + (d+ak+bh)(124)\\
&= f(211) + (af^2+fh)(222) + cf^2(123) + (cfg+df^2)(124)\\
&\qquad + g(212) + cf^2g(124) + h(112) + f^2h(222) + (h^2+fh^2)(123)\\
&\qquad + (hk+fhk)(124) + k(113) + kh(f^2+f+1)(124) + l(114).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&af(f-1) + hf(f+1) = 0,\\
&c(f^2-1) + h^2(f+1) - ah = 0,\\
&cgf(f+1) + d(f^2-1) + kh(f^2+2f+2) - ak - bh = 0.
\end{aligned}
$$

From i^3 we find

$$(a^2+c)c(f^2-1) + ach(f^2+f+1) = 0.$$

From j^2i we find

$$c(f^4-1) - ah(f+1) + h^2(f+1)(f^2+f+1) = 0.$$

This yields twelve sub-types:

(1) $c=0=a=d, \quad h\neq 0, \quad f=-1, \quad b=k.$

$$
\begin{aligned}
i^2 &= bij^2 + ej^4;\\
ji &= -ij + gij^2 + hj^2 + bj^3 + lj^4.
\end{aligned}
$$

(2) $c = 0 = h = k, \quad a \neq 0, \quad f = 1.$
$i^2 = aij + bij^2 + dj^3 + ej^4;$
$ji = ij + gij^2 + lj^4.$

(3) $c = 0 = f, \quad h = a \neq 0, \quad d = a(k - b).$
$i^2 = aij + bij^2 + a(k - b)j^3 + ej^4;$
$ji = gij^2 + aj^2 + kj^3 + lj^4.$

(4) $c = 0 = a = h = d.$
$i^2 = bij^2 + ej^4;$
$ji = fij + gij^2 + kj^3 + lj^4.$

(5, 6) $c = 0 = a = h, \quad f = \pm 1, \quad d \neq 0.$
$i^2 = bij^2 + dj^3 + ej^4.$
$ji = \pm ij + gij^2 + kj^3 + lj^4.$

(7) $a = 0, \quad h \neq 0, \quad f = -1, \quad d = 0, \quad b = k.$
$i^2 = bij^2 + cj^2 + ej^4;$
$ji = -ij + gij^2 + hj^2 + bj^3 + lj^4.$

(8) $a \neq 0, \quad h = 0, \quad f = 1, \quad g = 0, \quad a \neq 2\sqrt{-c}.$
$i^2 = aij + bij^2 + cj^2 + dj^3 + ej^4;$
$ji = ij + kj^3 + lj^4.$

(9) $a \neq 0, \quad h = 0, \quad f = 1, \quad g \neq 0, \quad a = 2\sqrt{-c}.$
$i^2 = 2\sqrt{-c}\, ij + bij^2 + cj^2 + dj^3 + ej^4;$
$ji = ij + gij^2 + kj^3 + lj^4.$

(10) $a = 0 = h, \quad f = -1.$
$i^2 = bij^2 + cj^2 + dj^3 + ej^4;$
$ji = -ij + gij^2 + kj^3 + lj^4.$

(11) $a = 0 = h = g = k, \quad f = 1.$
$i^2 = bij^2 + cj^2 + dj^3 + ej^4;$
$ji = ij + lj^4.$

(12) $h = \pm\sqrt{-1}\, a \neq 0, \quad f = \mp\sqrt{-1}, \quad c = -\tfrac{1}{2} a^2,$
$b = \tfrac{1}{2} ag(1 - \sqrt{-1}), \quad d = \tfrac{1}{2} a^2 g + \tfrac{1}{2} ak(1 - \sqrt{-1}).$
$$i^2 = aij + \tfrac{1}{2} ag(1 - \sqrt{-1})\, ij^2 - \tfrac{1}{2} a^2 j^2 + (\tfrac{1}{2} a^2 g + \tfrac{1}{2} ak(1 - \sqrt{-1}))\, j^3 + ej^4;$$
$ji = \mp\sqrt{-1}\, ij + gij^2 \pm \sqrt{-1}\, aj^2 + kj^3 + lj^4.$

(*e*) *When* $r = 9$.

9. $$\begin{aligned} i &= (210) + a(221) + b(222) + c(123) + d(124) + e(125); \\ j &= (111) + f(221) + g(222) + h(123) + k(124) + l(125); \\ ij &= (211) + af(222) + cf(124) + (cg + df)(125); \\ j^2 &= (112) + f^2(222) + h(1+f)(124) + (k + kf + hg)(125); \\ ij^2 &= (212) + cf^2(125); \\ j^3 &= (113) + h(f^2 + f + 1)(125). \end{aligned}$$

Then

$$\begin{aligned} i^2 = a(211) + a^2(222) + ac(124) + (bc + ad)(125) \\ + b(212) + c(113) + d(114). \end{aligned}$$

Hence

$$\begin{aligned} &a^2(f-1) = 0, \quad ac(f-1) = 0, \\ &agc + ad(f-1) + bc(f^2-1) + ch(f^2+f+1) = 0. \end{aligned}$$

Again

$$\begin{aligned} ji = f(211) + af(222) + c(124) + d(125) \\ + h(113) + ah(124) + bh(125) + k(114) + ak(125). \end{aligned}$$

Hence

$$\begin{aligned} &af(f-1) = 0, \quad af^2c - c - ah = 0, \\ &cfg + f^2d + h^2(f^2+f+1) = d + bh + ak. \end{aligned}$$

Again

$$i^3 = a^2(212) + a^2c(125) + ac(114) + (bc + ad)(115).$$

Hence

$$a^2c(f^2-1) = 0.$$

Again

$$j^2i = f^2(212) + (c + ah + afh)(125) + h(f+1)(114) + k(f+1)(115).$$

Hence

$$cf^4 = afh + ah + c.$$

These equations reduce to

$$\begin{aligned} &a(f-1) = 0, \\ &agc + bc(f^2-1) + ch(f^2+f+1) = 0, \\ &af^2c - c - ah = 0, \\ &c(f^4-1) = 0, \\ &fgc + f^2d + h^2(f^2+f+1) - d - bh - ak = 0. \end{aligned}$$

The sub-types are

(1) $a = 0 = c, \quad f = 1, \quad h = 0.$

$$\begin{aligned} i^2 &= bij^2 + dj^4 + ej^5; \\ ji &= ij + kj^4 + lj^5. \end{aligned}$$

(2) $a = 0 = c,\quad f = 1,\quad h \neq 0,\quad b = 3h.$
$i^2 = 3hij^2 + dj^4 + ej^5;$
$ji = ij + hj^3 + kj^4 + lj^5.$

(3) $a \neq 0,\quad f = 1,\quad c = 0 = h = k.$
$i^2 = aij + bij^2 + dj^4 + ej^5;$
$ji = ij + lj^5.$

(4) $a \neq 0 \neq g,\quad f = 1,\quad c = -\dfrac{a^2g}{3(a-1)},$

$h = -\tfrac{1}{3}\,ag,\quad k = -\dfrac{ag^2}{3(a-1)} + g\,(3ag + b).$

$i^2 = aij + bij^2 - \dfrac{a^2g}{3(a-1)}\,j^3 + dj^4 + ej^5;$

$ji = ij - \tfrac{1}{3}\,agj^3 + \left(\dfrac{-ag^2}{3(a-1)} + g\,(3ag + b)\right)j^4 + lj^5.$